One parameter family of indecomposable optimal entanglement witnesses arising from generalized Choi maps

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In the recent paper [Chruściński and Wudarski, arXiv:1105.4821], it was conjectured that the entanglement witnesses arising from some generalized Choi maps are optimal. We show that this conjecture is true. Furthermore, we show that they provide a one parameter family of indecomposable optimal entanglement witnesses.

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Quantum entanglement is a basic resource in quantum information processing and communication [1]. Therefore, so much effort naturally has been put into developing theoretical and experimental methods of entanglement detection. Among them, one of the most general approach is based on the notion of entanglement witness [2, 3]. Recall that an observable $W = W^\dagger$ is said to be an entanglement witness (EW) if $\mathrm{tr}(W\sigma) \geq 0$ for all separable states σ, and there exists an entangled state ρ for which $\mathrm{tr}(W\rho) < 0$. In this case we say that W detects ρ. Following [4], an EW W is said to be optimal if the set of entanglement detected by W is maximal with respect to the set inclusion.

Note that the Choi-Jamiołwski isomorphism [5, 6] gives rise to an entanglement witness

$$W_\Lambda = (\mathbb{1} \otimes \Lambda)P^+,$$

which is acting on $M_M \otimes M_N$, for every positive linear map $\Lambda : M_M \to M_N$ which is not completely positive, where M_K denotes the C^*-algebra of all $K \times K$ matrices over the complex field $\mathbb{C}$, and P^+ denotes the projector onto the maximally entangled state in $\mathbb{C}^M \otimes \mathbb{C}^M$. It is well known that decomposable positive maps give decomposable entanglement witnesses which take the general form $W = P + Q^\Gamma$, where P, $Q \geq 0$ and Γ refers to partial transposition with respect to the second subsystem, that is, $Q^\Gamma = (\mathbb{1} \otimes T)Q$. If a given witness can not be written in this form, we call it indecomposable. Of course, indecomposable EWs arise from indecomposable positive maps [4, 7, 8]

Note that an EW is indecomposable if and only if it detects entangled states with positive partial transposes [4]. Thus, so far indecomposable EWs are concerned, it is natural to consider the optimality by requiring the witness to be finer with respect to entangled states with positive partial transposes only. This kinds of witness is said to be an indecomposable optimal EW. It is known [4, 9] that W is an indecomposable optimal entanglement witness if and only if both W and W^Γ are optimal entanglement witnesses.

Typical examples of indecomposable optimal EW come from indecomposable positive linear maps which generate an extremal ray of the convex cone consisting of all positive linear maps. The Choi map is an example of this kind [5, 9, 10]. Variations of the Choi map given by the second author [11] also give rise to such maps. Some of them, parameterized by three real variables, were shown to be extremal in [12]. See the recent paper [13] for related topics. Although there are some examples of optimal EW [9, 14–17], to the best of the author's knowledge, only known examples of indecomposable optimal EWs are ones which come from extremal indecomposable positive linear maps.

We consider another variations of the Choi map given in [18]. For nonnegative real numbers a, b, c, we define the linear map $\Phi[a, b, c] : M_3 \to M_3$ by

$$\Phi[a,b,c](X) = \frac{1}{a+b+c}\begin{pmatrix} ax_{11}+bx_{22}+cx_{33} & -x_{12} & -x_{13} \\ -x_{21} & cx_{11}+ax_{22}+bx_{33} & -x_{23} \\ -x_{31} & -x_{32} & bx_{11}+cx_{22}+ax_{33} \end{pmatrix},$$

for $X = [x_{ij}] \in M_3$. It was shown that $\Phi[a, b, c]$ is positive if and only if

$$a+b+c \geq 2, \qquad 0 \leq a \leq 2 \implies bc \geq (1-a)^2.$$

In the case of $a = 1$, the maps $\Phi[1, 1, 0]$ and $\Phi[1, 0, 1]$ reproduce the Choi map and its dual, respectively [5]. The matrix representation of EW $W[a, b, c] := W_{\Phi[a,b,c]}$

is given by

$$W[a,b,c]=\frac{1}{6}\left(\begin{array}{ccc|ccc|ccc} a & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & -1 \\ \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & c & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot \\ -1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & a \end{array}\right),$$

where we replaced zeros by dots.

Recently, Chruściński and Wudarski [17] analyzed the following case:

$$0 \le a \le 1, \quad a+b+c=2, \quad bc=(1-a)^2, \tag{1}$$

and parameterize them by

$$\begin{aligned} a(\alpha) &= \frac{2}{3}(1+\cos(\alpha)), \\ b(\alpha) &= \frac{2}{3}\left(1-\frac{1}{2}\cos(\alpha)-\frac{\sqrt{3}}{2}\sin(\alpha)\right), \\ c(\alpha) &= \frac{2}{3}\left(1-\frac{1}{2}\cos(\alpha)+\frac{\sqrt{3}}{2}\sin(\alpha)\right), \end{aligned}$$

for $\pi/3 \le \alpha \le 5\pi/3$. They conjectured that the entanglement witnesses $W[\alpha]=W[a(\alpha),b(\alpha),c(\alpha)]$ arising from them are optimal for every α with $\pi/3 \le \alpha \le 5\pi/3$.

The purpose of this Brief Report is to show that this conjecture is true. Furthermore, we show that $W[\alpha]$ is an indecomposable optimal EW for each $\pi/3 < \alpha < \pi$ and $\pi < \alpha < 5\pi/3$. Equivalently, we show the following:

Theorem 1 *If a, b, c are nonnegative numbers satisfying the conditions* (1) *then entanglement witnesses $W[a,b,c]$ are optimal. Furthermore, $W[a,b,c]$ is an indecomposable optimal entanglement witness, whenever $0<a<1$.*

For an EW W, define

$$\mathcal{P}_W=\{\psi\otimes\phi\in\mathcal{H}_A\otimes\mathcal{H}_B \,:\, \langle\psi\otimes\phi|W|\psi\otimes\phi\rangle=0\}.$$

It is known [4] that if the set $\mathcal{P}_W$ spans the entire Hilbert space $\mathcal{H}_A\otimes\mathcal{H}_B$, then W is an optimal EW.

Proof: It suffices to consider the case $0 \le a < 1$. Put $t=c/(1-a)$ for a, c satisfying the condition (1). Then t is a positive number and satisfies the following conditions:

$$a+bt=1, \qquad c+at=t. \tag{2}$$

For each $k=1,2,\cdots,9$, we define vectors $|\psi_k\rangle$, $|\phi_k\rangle$ in $\mathbb{C}^3$ as follows:

$$\begin{aligned} |\psi_1\rangle &= |0\rangle+|1\rangle+|2\rangle, & |\phi_1\rangle &= |0\rangle+|1\rangle+|2\rangle, \\ |\psi_2\rangle &= |0\rangle-|1\rangle+|2\rangle, & |\phi_2\rangle &= |0\rangle-|1\rangle+|2\rangle, \\ |\psi_3\rangle &= |0\rangle+i|1\rangle-i|2\rangle, & |\phi_3\rangle &= |0\rangle-i|1\rangle+i|2\rangle, \\ |\psi_4\rangle &= \sqrt{t}|1\rangle+|2\rangle, & |\phi_4\rangle &= \sqrt{t}|1\rangle+t|2\rangle, \\ |\psi_5\rangle &= \sqrt{t}|1\rangle+i|2\rangle, & |\phi_5\rangle &= \sqrt{t}|1\rangle-ti|2\rangle, \\ |\psi_6\rangle &= |0\rangle+\sqrt{t}|2\rangle, & |\phi_6\rangle &= t|0\rangle+\sqrt{t}|2\rangle, \\ |\psi_7\rangle &= i|0\rangle+\sqrt{t}|2\rangle, & |\phi_7\rangle &= -ti|0\rangle+\sqrt{t}|2\rangle, \\ |\psi_8\rangle &= \sqrt{t}|0\rangle+|1\rangle, & |\phi_8\rangle &= \sqrt{t}|0\rangle+t|1\rangle, \\ |\psi_9\rangle &= \sqrt{t}|0\rangle+i|1\rangle, & |\phi_9\rangle &= \sqrt{t}|0\rangle-ti|1\rangle. \end{aligned}$$

For $k=1,2,3$, it is easy to see that

$$\begin{aligned} &\langle\psi_k\otimes\phi_k|W[a,b,c]|\psi_k\otimes\phi_k\rangle \\ =&\langle\psi_k\otimes\phi_k^*|(W[a,b,c])^\Gamma|\psi_k\otimes\phi_k^*\rangle \\ =&3(a+b+c-6)=0. \end{aligned}$$

Using the condition (2), we also have

$$\begin{aligned} &\langle\psi_k\otimes\phi_k|W[a,b,c]|\psi_k\otimes\phi_k\rangle \\ =&\langle\psi_k\otimes\phi_k^*|(W[a,b,c])^\Gamma|\psi_k\otimes\phi_k^*\rangle \\ =&bt^3+2(a-1)t^2+ct \\ =&(at+c)t+(a+bt)t^2-2t^2, \\ =&t^2+t^2-2t^2=0 \end{aligned}$$

for $k=4,5,\cdots,9$. So all the vectors $\psi_k\otimes\phi_k$ ($\psi_k\otimes\phi_k^*$ respectively) belong to the set $\mathcal{P}_{W[a,b,c]}$ ($\mathcal{P}_{(W[a,b,c])^\Gamma}$ respectively).

Now, we show that the set

$$\{\psi_k\otimes\phi_k : 1\le k\le 9\} \tag{3}$$

spans the entire space $\mathbb{C}^3\otimes\mathbb{C}^3$ for $0\le a<1$, and the set

$$\{\psi_k\otimes\phi_k^* : 1\le k\le 9\} \tag{4}$$

spans the entire space for $0<a<1$. Let M (M' respectively) be the 9×9 matrix whose k-th column is $|\psi_k\otimes\phi_k\rangle$ ($|\psi_k\otimes\phi_k^*\rangle$ respectively). Then the determinant of M and M' are given by

$$\begin{aligned} |M| =& 8t^4(t^2-1)\sqrt{t}(2t-\sqrt{t}+2) \\ &-8t^5(1+t)(t-4\sqrt{t}+1)i, \\ |M'| =& -8t^4\sqrt{t}(t-1)^3-8t^4\sqrt{t}(t-1)^3 i. \end{aligned}$$

The only positive solution of the equation $\mathrm{Re}(|M|)=0$ is $t=1$, where $\mathrm{Re}(z)$ denotes the real part of complex number z. But for this $t=1$, we have $|M|\ne 0$. Therefore we can conclude that $|M|\ne 0$ for positive number t. Consequently, the set (3) spans the entire space, and so $W[a,b,c]$ is an optimal EW.

On the other hand, the only positive solution of the equation $|M'|=0$ is $t=1$. Since $t=1$ is equivalent

to $(a,b,c)=(0,1,1)$, the set (4) spans the entire space except for the case $(a,b,c)=(0,1,1)$. Therefore, we conclude that $W[a,b,c]$ is an indecomposable optimal EW except for the case of $a=0$.

When $a=0$, $\Phi[0,1,1]$ is a decomposable positive map, and the optimality of $W[0,1,1]=W[\pi]$ is known in [17]. This completes the proof of Theorem 1.

In conclusion, we proved that the conjecture posed by Chruściński and Wudarski is true. Moreover, we provided a one parameter family of indecomposable optimal EWs. In [17], Chruściński and Wudarski showed that the structural physical approximation defines a separable state for a large class of $W[\alpha]$. So our result supports the another conjecture [9] that the SPA to an optimal entanglement witness defines a separable state. Finally, it woul be interesting to know if the map $\Phi[a,b,c]$ generates an extremal ray of the cone of all positive linear maps whenever the condition (1) holds.

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